Long Time Existence of Non-Compact Inverse Mean Curvature Flow in Hyperbolic Space

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Abstract

We investigate Inverse Mean Curvature Flow (IMCF) of non-compact hypersurfaces in hyperbolic space. Specifically, we look at bounded graphs over horospheres in $\mathbb{H}^{n+1}$ and show long time existence of the flow. Along the way many important local estimates as well as global estimates are obtained. In addition, we develop a useful family of cutoff functions for IMCF as well as a non-compact ODE maximum principle at infinity which are integral tools used throughout the document.

1. Introduction:

We will investigate the geometric evolution of hypersurfaces Σ^n through a one parameter family of embeddings $\varphi : \Sigma \times [0,T) \to N^{n+1}$, φ satisfying inverse mean curvature flow

$$\begin{cases} \frac{\partial \varphi}{\partial t}(p,t) = \frac{\nu(p,t)}{H(p,t)} & \text{for } (p,t) \in \Sigma \times [0,T) \\ F(p,0) = \Sigma_0 & \text{for } p \in \Sigma \end{cases} \tag{1}$$

where H is the mean curvature of $\Sigma_t := \varphi_t(\Sigma)$ and ν is a consistently chosen normal vector (we will be more specific later). In this paper we will specifically investigate $N = \mathbb{R}^{n+1}$ or $N = \mathbb{H}^{n+1}$ and we will require Σ_0 to be represented as a graph over a cylinder or a plane, respectively.

IMCF was used by R. Geroch [11] in his approach to the Penrose Inequality. He successfully implemented this approach in the rotationally symmetric case but the general case was left open. Global existence results for non-symmetric initial hypersurfaces in euclidean space were obtained several years later by Gerhardt [8] and Urbas [25] who independently proved that any compact, mean-convex and star-shaped hypersurface will asymptotically approach a sphere and converge to a sphere after an appropriate rescaling under IMCF (as well as a whole family of inverse flows).

Since then there have been extensions of this theorem to Lorentzian manifolds [9], hyperbolic space [10] [3] as well as to rotationally symmetric spaces with non-positive radial curvature [23]. There has also been a great deal of work on weak solutions of IMCF including viscosity solutions [2], weak solutions through connection to the p-Laplacian [19] as well as the most famous formulation of weak variational solutions to IMCF by Huisken and Ilmanen [15] which were used to prove the Riemannian Penrose Inequality (time symmetric case) and hence fulfill the program laid out by Geroch in the rotationally symmetric case.

The development of IMCF for non-compact hypersurfaces inside Riemannian manifolds has not seen much consideration compared to the compact case but there are important and well understood results for non-compact Mean Curvature Flow (MCF) that should be mentioned. Ecker and Huisken [5] were able to show convergence to a translating soliton for graphs over planes, satisfying certain initial growth conditions, in $\mathbb{R}^{n+1}$ under MCF by using a maximum principle which follows from Huisken's monotonicty formula. Later, they developed further interior estimates for non-compact MCF [6] as well as a non-compact maximum principle that works for a fairly general class of evolution equations with time dependent metrics including Ricci Flow.

The results of Ecker and Huisken were extended by Rasul in his dissertation thesis [22], advised by Ecker, where he was able to relax the initial growth conditions at infinity and show "slow" convergence to a self-similar solution. Non-compact MCF was further extended to radial graphs in $\mathbb{H}^{n+1}$ by Untenberger [24], who was able to show long time existence for locally Lipschitz entire radial graphs (graphs over $S^n_+ \subset \mathbb{H}^{n+1}$ parameterized using the upper half space model) and also convergence to a hyperplane S^n_+ in the upper half space model of $\mathbb{H}^{n+1}$ for entire radial graphs with bounded hyperbolic height.

The non-compact case of IMCF has seen almost no attention besides the specific examples given by Huisken and Illmanen [14] and the recent paper on solitons of IMCF by Drugan, Lee and Wheeler [4]. Besides these examples of special solutions there has been no work on showing convergence to a prototypical hypersurface for a class of initial data as has been done for compact IMCF for the sphere. The present work changes this by studying non-compact IMCF in Hyperbolic space. In this paper we are concerned with long time existence of IMCF in $\mathbb{H}^{n+1}$ and more precisely we prove the following theorem

Theorem 1. *Let Σ_t be a smooth solution of IMCF with initial hypersurface Σ_0 satisfying the following bounds $0 < H_0 \leq H(x,0) \leq H_1 < \infty$ and $|A|(x,0) \leq A_0 < \infty$. We further assume that Σ_0 can be represented as a graph of a bounded function with bounded gradient, over and uniformly bounded away from $\mathbb{R}^n \times \{0\}$ in the upper half space model of hyperbolic space. Then the IMCF starting at Σ_0 exists for all time $t \in [0, \infty)$.*

In the second section we start by stating and discussing the proof of short time existence for the flow. Then we obtain global estimates as well as a continuation criterion which lead to long time existence. This section ends by finishing the proof of Theorem 1 which relies on Theorem 7 for a lower bound on H. In this section we show how to complete the proof of Theorem 1 assuming this lower bound and then use the rest of the paper to find this lower bound as well as other local estimates for the flow. We also briefly discuss what we expect as far as asymptotic properties of the flow (work in progress).

In the third section we develop techniques for IMCF which have been important in MCF [7] that allow us to compute evolution equations on our evolving hypersurface for functions defined in the ambient space which we call extrinsically defined functions. We do this in a fairly general setting and to our knowledge these techniques have not been used for IMCF. This section culminates with the construction of cutoff functions which satisfy useful evolution equations.

In the fourth section we start by discussing important notation and then we compute evolution equations for important geometric quantities as well as take the previously constructed cutoff functions and obtain the first estimate for the support function.

In the fifth section we continue using cutoff functions to obtain important detailed local estimates for the mean curvature and the second fundamental form. This section ends with the proof of Theorem 5 where we unpack all of the local estimates obtained in this section and state them in an organized way for important geometric quantities related to the admissible set of Theorem 1. Then we state and prove Corollary 3 which shows that the local estimates obtained in this section extend to the case where $R \to \infty$ which gives us the necessary lower bound on H needed to complete the proof of Theorem 1.

In the appendix we include a statement and proof of an ODE maximum principle at infinity which allows us to use the Omari-Yau maximum principle at infinity [1], [20], [26], [21] to extend the ODE maximum principle of Hamilton [12], [18] to the case of bounded (in space) functions defined on non-compact domains. This result is used several times throughout the document and to our knowledge has not been previously proven.

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2. Short Time and Long Time Existence:

In the following theorem we are aiming to use a similar argument to [9] in order to show short time existence so we will not write down the whole argument from scratch. Rather we will use this to fix our notation for how we are expressing the equation (1) in terms of a graph function over a plane or cylinder, as well as note the ellipticity of the corresponding operator.

Theorem 2. ***Short Time Existence :*** *If $\Sigma_0 \subset \mathbb{H}^{n+1}$ is a hypersurface to which the hypotheses of Theorem 1 apply then there is a solution to (1) for all $t \in [0, \epsilon)$ where ϵ depends on the initial data.*

Proof. In this section we will use bars to denote quantities w.r.t. $\mathbb{H}^{n+1}$, no bars for quanitities w.r.t Σ_t and subscript 0 for quantities w.r.t. $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$.

First we notice that the following flow

$$\left(\frac{\partial \varphi}{\partial t}\right)^{\perp} = \frac{\nu}{H} \tag{2}$$

is , up to tangential diffeomorphisms, equivalent to IMCF. Now if we write Σ_t as a graph over $\{y = 0\}$ then we have the expressions $\varphi(p,t) = (x, y(x,t))$ and $\bar{\nu} = \frac{y(\nabla^0 y, -1)}{\sqrt{1+|\nabla^0 y|^2}}$. Now we notice that

$$\bar{g}\left(\frac{\partial \varphi}{\partial t}, \nu\right) = \frac{-1}{y\sqrt{1+|\nabla^0 y|^2}} \frac{\partial y}{\partial t} = \frac{1}{H} \quad \Rightarrow \quad \frac{\partial y}{\partial t} = \frac{-y\sqrt{1+|\nabla^0 y|^2}}{H} = \frac{-vy}{H}$$

where we have used the fact that $v := \sqrt{1+|\nabla^0 y|^2}$.

Now if we use that fact that $H = \frac{n + y\tilde{\delta}^{ij} y_{ij}}{v}$ then we find

$$\frac{\partial y}{\partial t} = \frac{-yv^2}{n + y\tilde{\delta}^{ij} y_{ij}} = F(x, y, \nabla^0 y, \nabla^0 \nabla^0 y)$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \to \mathbb{R}$, denoted $F(z, u, p_i, a_{ij})$, from which we notice

$$\frac{\partial F}{\partial a_{ij}} = \frac{yv^2}{(n + y\tilde{\delta}^{ij} y_{ij})^2} y\tilde{\delta}^{ij}\delta_{ij} = \frac{y^2}{H^2}\tilde{\delta}^{ij}\delta_{ij}$$

so if our initial condition $y(x,0) \in \{y \in C^2(\mathbb{R}^n) : 0 < H_1 < H(x,0) < H_2 < \infty, v(x,0) \leq v_0 < \infty$ and $0 < y_0 < y(x,0) \leq y_1 < \infty\}$ then we have that $\frac{\partial F}{\partial a_{ij}} \geq c_0 \frac{y_0^2}{H_2^2}$ and also bounded and so the linearized operator is uniformly parabolic and hence a solution exists for a short time. See [9] for more details in the compact case. ∎

Now we look to obtain some further estimates that will culminate with a long time existence result. We start out with a concrete example of the evolution of horospheres in $\mathbb{H}^{n+1}$ and then we show that horospheres act as barriers in $\mathbb{H}^{n+1}$ for hypersurfaces satisfying the hypotheses of Theorem 1.

Example: Consider the horosphere $y = y_0$ as a graph over $\mathbb{R}^n \times \{0\}$. Then y is just a function of time and $H = n$ and so we find the ODE

$$\frac{dy}{dt} = \frac{-y}{n}$$

which has the solution $y(t) = y_0 e^{-t/n}$.

The following two Theorems will demonstrate that the above examples act as barriers.

Theorem 3. *If $0 < \inf_{\mathbb{R}^n} y(x,0) = y_0$ and $\sup_{\mathbb{R}^n} y(x,0) \leq y_1$ and we assume that Σ_0 is a hypersurface to which the hypotheses of Theorem 1 apply then we find that*

$$y_0 e^{-t/n} \leq y(x,t) \leq y_1 e^{-t/n}$$

So horospheres act as barriers for bounded graphs over $\mathbb{R}^n$.

Proof. Notice that by construction, as well as by Theorem 2, the function $y(x,t)$ is bounded above and below and hence we can use the result that $y_{inf}(t) = \inf_{\mathbb{R}^n} y(x,t)$ is a well defined, locally Lipschitz function. Then if we let $\{x_k\} \in \mathbb{R}^n$ be a sequence so that $\lim_{k\to\infty} y(x_k,t) = \inf_{\mathbb{R}^n} y(x,t)$ then we know by the maximum principle at infinity that

$$|\nabla^0 y(x_k,t)| < \frac{1}{k} \qquad \nabla^0\nabla^0 y(x_k,t) > -\frac{1}{k}\delta$$

and so if we use the expressions for H and w in terms of graphs we find

$$H = \frac{n + y\tilde{\delta}^{ij} y_{ij}}{\sqrt{1+|\nabla^0 y|^2}} \quad \Rightarrow \quad H(x_k,t) \geq \frac{n - k^{-1} y\tilde{\delta}^{ij}\delta_{ij}}{\sqrt{1+\frac{1}{k^2}}} \quad \Rightarrow \quad \lim_{k\to\infty} H(x_k,t) \geq n$$

$$w = \frac{1}{y\sqrt{1+|\nabla^0 y|^2}} \quad \Rightarrow \quad w(x_k,t) = \frac{1}{y(x_k,t)\sqrt{1+|\nabla^0 y(x_k,t)|^2}} \quad \Rightarrow \quad \lim_{k\to\infty} w(x_k,t) = \frac{1}{y_{inf}(t)}$$

Now since we have the following ODE for $y(x,t)$

$$\frac{\partial}{\partial t}\left(\frac{1}{y^2}\right) = \frac{\partial}{\partial t}\bar{g}(\partial_y,\partial_y) = \frac{2}{H}\bar{g}(\bar{\nabla}_{\bar{\nu}}\partial_y,\partial_y) = \frac{2}{H}\bar{g}(-\frac{\bar{\nu}}{y},\partial_y) = \frac{2}{yH}\bar{g}(\bar{\nu},\eta) = \frac{2w}{yH}$$

we can find

$$\frac{\partial}{\partial t}\left(\frac{1}{y_{inf}^2}\right) \leq \frac{2}{n y_{inf}^2}$$

Then by integrating we find

$$\frac{1}{y_{inf}^2} \leq \max_{\mathbb{R}^n \times \{0\}} \frac{1}{y_{inf}^2} e^{2t/n} = \frac{1}{y_0^2} e^{2t/n}$$

which yields the desired estimate.
Using a similar argument for $y_{sup}(t) = \sup_{\mathbb{R}^n} y(x,t)$ we find the other important estimate. □

Our last goal is to extend the short time uniform bounds from Theorem 2 to give us long time existence. We start this goal by obtaining upper bounds on $|A|^2$ in terms of bounds on H by using the ODE maximum principle at infinity, which is stated and proven in the appendix, which will lead to a continuation criterion.

Theorem 4. *Let Σ_0 be a hypersurface to which the hypotheses of Theorem 1 apply. Then consider the tensor $M_{ij} = HA_{ij}$ where $\{\kappa_1, ..., \kappa_n\}$ are the eigenvalues of M_{ij} and $\{\lambda_1, ..., \lambda_n\}$ are the eigenvalues of A_{ij}. Then we have the following estimates for these eigenvalues*

$$\kappa_i \leq \frac{nH_1^2}{H_0^2} \cdot \frac{e^{\frac{2nt}{H_0^2}}}{C + e^{\frac{2nt}{H_0^2}}} \qquad \lambda_i \leq \frac{n}{H_0} \cdot \frac{H_1^2}{H_0^2} \cdot \frac{e^{\frac{2nt}{H_0^2}}}{C + e^{\frac{2nt}{H_0^2}}} \qquad \text{for } \mathbb{H}^{n+1}$$

for all $t \in [0,T)$ where $C = \frac{nH_1^2}{C_0 H_0^2}$ and $C_0 = \sup_{x \in \mathbb{R}^n} \kappa_{\max}(x,0)$.

Proof. We consider the evolution equation for M_{ij} and use the fact that, if we choose a o.n. frame w.r.t $\mathbb{H}^{n+1}$, we can find

$$\bar{R}_{ijkl} = \delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}, \quad \bar{R}_{ij} = -n\delta_{ij}, \quad \bar{R} = -n(n+1)$$

which leads to

$$\left(\partial_t - \frac{1}{H^2}\Delta\right)(M_i^j) \leq -2\frac{\nabla_i H \nabla^j H}{H^2} - 2\frac{1}{H^3}\nabla^k M_i^j \nabla_k H - 2\frac{M^{jk}M_{ik}}{H_1^2} + \frac{2n}{H^2}M_i^j$$

see Lemma 3 below for the relevant evolution equation.

Now if we let $\kappa(x,t)$ be the maximum eigenvalue of $M_i^j(x,t)$ and then define $\kappa_{sup} = \sup_{\Sigma_t}\kappa(x,t) < \infty$ then we know that κ_{sup} is locally lipschitz and satisfies the following ODE

$$\frac{\partial \kappa_{sup}}{\partial t} \leq \frac{-2}{H_1^2}\kappa_{sup}^2 + \frac{2n}{H_0^2}\kappa_{sup} = \frac{2\kappa_{sup}}{H_1^2}\left(\frac{nH_1^2}{H_0^2} - \kappa_{sup}\right)$$

from which we can find using the ODE maximum principle at infinity [See Appendix], which we know applies by Theorem 2 since $Rc = \bar{R}c - HA + A^2$ which means that $|Rc| \leq (n + H|A| + |A|^2) \leq C$, the following bound

$$\kappa_{sup}(t) \leq \frac{nH_1^2}{H_0^2}\left(\frac{e^{\frac{2nt}{H_0^2}}}{C + e^{\frac{2nt}{H_0^2}}}\right)$$

□

Note: After we obtain various local estimates in Theorem 6 and turn them into global estimates in Corollary 3, both at the end of section 5, the above theorem is not necessary since we will have obtained an upper bound on $|A|^2$. We include this theorem anyway since it holds for any IMCF with upper and lower bounds on H and lower bound on Ricci curvature (or any other condition which implies that the ODE maximum principle at infinity applies).

Theorem 5. *If we assume that Σ_0 is a hypersurface to which the hypotheses of Theorem 1 apply then we find*

$$H(x,t) \leq \sqrt{c_0 e^{-2t/n} + n^2} \qquad \textit{for } \mathbb{H}^{n+1}$$

where $c_0 = H_{sup}(0)^2 - n^2$ if $H_{sup}(0) > n$ and $c_0 = 0$ if $H_{sup}(0) \leq n$.

Proof. We have the evolution equation for H

$$(\partial_t - \frac{1}{H^2}\Delta)H = -2\frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} + \frac{n}{H}$$

and by Theorem 2 we know that H is bounded above for a short time t and by the note given in Theorem 5 we can use the ODE maximum principle at infinity [See Appendix] on the following ODE

$$\frac{dH_{sup}}{dt} \leq \frac{1}{nH_{sup}} \left(n^2 - H_{sup}^2\right)$$

from which it follows that $H_{sup}(t) \leq \sqrt{c_0 e^{-2t/n} + n^2}$ where $c_0 = H_{sup}(0)^2 - n^2$ if $H_{sup}(0) > n$ and $c_0 = 0$ if $H_{sup}(0) \leq n$. $\square$

Note: Again you will notice that we obtain upper bounds on H in Corollary 3 at the end of section 5 but the estimates above are cleaner in the global case.

Proof of Theorem 1

The proof of Theorem 1 is now finished in the exact same way as [16] because we have an upper bound on $H(x,t)$ for all time $t \in [0,T)$ from Theorem 5 and a lower bound on $H(x,t)$ from Corollary 3 in Section 5 and so we can use Theorem 4 to give us an upper bound on $|A|^2$ for all $t \in [0,T)$. Then if $T < \infty$ is the maximal time of existence we can use the fact that $|A|^2$ is bounded to bound all of its derivatives, see [8],[9],[17] and [25], hence extract a subsequence of times t_i so that $\Sigma_{t_i} \to \Sigma_T$ as $i \to \infty$ where Σ_T is a smooth hypersurface with the same uniform bounds on w and H. Then we can apply the short time existence results, Theorem 2, with initial condition Σ_T and hence continue the flow which contradicts the fact that T was supposed to be maximal.

Note: One can obtain the necessary lower bound on H by using the ODE maximum principle at infinity as well as the evoltultion equation for u, computed below, but we will obtain it through the local estimates that we obtain in section 5.

Corollary 1. *Let $\varphi : \Sigma \times [0,T) \to \mathbb{H}^{n+1}$ be a smooth solution of IMCF so that $0 < H_0 < H(x,0) < H_1 < \infty$. If $H(x,t)$ remains bounded below by some constant H_0' for all $t \in [0,T)$ then the flow can be continued beyond the time T.*

Proof. Since we know that for hypersurfaces, Σ_0 with $H(x,0) < H_1 < \infty$, Theorem 5 combined with short time existence will give us an upper bound on H we can apply the same argument above for the proof of Theorem 1 to show the above continuation criterion for the flow, similar to [16]. $\square$

3. Extrinsically Defined Cutoff Functions:

As noticed in the last section, the proof of Theorem 1 relies on a lower bound of H so now for the rest of the paper we move on to obtaining that lower bound on H as well as

other important local estimates for the flow. In order to be able to prove any local estimates we need to develop well suited cutoff functions for IMCF and so we develop those tools in this section.

We consider the Riemannian manifold N^{n+1} paramatrized over $\mathbb{R}_a^b := \{(x_1, ..., x_n, y) \in \mathbb{R}^{n+1} : a < y < b\}$ where $a, b \in [-\infty, \infty]$ with the metric $\bar{g} = \lambda(y)^2\delta$ where $\lambda : (a, b) \to \mathbb{R}$ is defined. We will consider a n dimensional, non-compact hypersurface $\Sigma_0 \subset \mathbb{R}_a^b$ of which we would like to be able to compute geometric quantities on Σ_0 in terms of covariant derivatives of N.

In this document we will use bars to denote geometric quantities w.r.t N^{n+1}, superscript 0 to denote quantities w.r.t. δ and no bar or subscript to denote quantities w.r.t. Σ_0.

By using well known formulas for conformal metrics we can find the following expression

$$\bar{\nabla}_X Y = \nabla_X^0 Y + \frac{\lambda'}{\lambda}\left(\langle X, \partial_y\rangle_0 Y + \langle \partial_y, Y\rangle_0 X - \langle X, Y\rangle_0 \partial_y\right)$$

Using this, and the convention that we will put a bar over a vector field $\bar{Z} = \lambda^{-1}(y)Z$ so that $\bar{Z}$ is a unit vector w.r.t. $\bar{g}$, we can obtain the following

$$\begin{aligned}
d\bar{i}vX &= \bar{g}(\bar{\nabla}_{\bar{e}_i}X, \bar{e}_i) \\
&= \langle \nabla_{e_i}^0 X + \frac{\lambda'}{\lambda}(\langle e_i, \partial_y\rangle_0 X + \langle \partial_y, X\rangle_0 e_i - \langle e_i, X\rangle_0 \partial_y, e_i\rangle_0 \\
&= div^0 X + (n+1)\frac{\lambda'}{\lambda}\langle X, \partial_y\rangle_0
\end{aligned}$$

where $\{e_1, ..., e_{n+1}\}$ is a orthonormal basis for $\mathbb{R}^{n+1}$ w.r.t the flat metric. From which we also obtain

$$H = -\frac{H_0}{\lambda} - n\frac{\lambda'}{\lambda^2}\langle \nu_0, \partial_y\rangle_0$$

Note: We are choosing the "downward" pointing normal which is important to keep in mind.

Now we can obtain a useful Proposition which will allow us to find many important evolution equations w.r.t IMCF.

Proposition 1. *Let $f : U \to \mathbb{R}$ where $U \subset \mathbb{R}^n \times \mathbb{R}_+$ open, we consider the function $g : \Sigma \times [0, T) \to \mathbb{R}$ defined by $g(p, t) = f(\varphi(p, t))$ which has the following evolution equation under IMCF*

$$(\partial_t - \frac{1}{H^2}\Delta^{\Sigma_t})g = \frac{1}{\lambda^2 H^2}\left(\langle \nabla_\nu^0 \nabla^0 f, \nu\rangle - \Delta^0 f - (n-2)\frac{\lambda'}{\lambda}\langle \nabla^0 f, \partial_y\rangle - 2\frac{\lambda'}{\lambda}\langle \nabla^0 f, \nu\rangle\langle \nu, \partial_y\rangle\right) + \frac{2}{\lambda H}\nabla_\nu^0 f$$

Proof. For any function u and vector field X we have that

$$\bar{div}(uX) = u\ \bar{div}(X) + X(u)$$
$$\bar{\nabla} u = \lambda^{-2}\nabla^0 u$$

Then we notice that

$$\partial_t g = \frac{1}{H}\bar{\nabla}_{\bar{\nu}} f$$
$$\Delta^{\Sigma_t} g = div(\nabla g) = div(\bar{\nabla} f - \bar{\nabla}_{\bar{\nu}} f \bar{\nu}) = div(\bar{\nabla} f) - H\bar{\nabla}_{\bar{\nu}} f$$

Note: Here is where we see a big difference between MCF and IMCF. When studying MCF there is a cancelation between the time derivative term and the first order term in the Laplacian which simplifies computations. In IMCF these two terms combine and hence give an extra term to deal with.

Now we can find the following expression where our goal is to first write all derivatives as extrinsic derivatives in N^{n+1} and then convert all of those derivatives to derivatives on $\mathbb{R}^{n+1}$, using the formulas obtained above.

$$\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta^{\Sigma_t})g &= \frac{2}{H}\bar{\nabla}_{\bar{\nu}} f - \frac{1}{H^2} div(\bar{\nabla} f) \\
&= \frac{1}{H^2}\left(-\bar{div}(\bar{\nabla} f) + \langle \bar{\nabla}_{\nu}\bar{\nabla} f, \nu\rangle_0\right) + \frac{2}{\lambda H}\nabla^0_{\nu} f \\
&= \frac{1}{H^2}\left(-\lambda^{-2}\bar{div}(\nabla^0 f) - \nabla^0 f(\lambda^{-2}) + \lambda^{-2}\langle \bar{\nabla}_{\nu}\nabla^0 f, \nu\rangle + \nu(\lambda^{-2})\langle \nabla^0 f, \nu\rangle\right) + \frac{2}{\lambda H}\nabla^0_{\nu} f \\
&= \frac{1}{H^2}\left(-\lambda^{-2}\Delta^0 f - (n+1)\frac{\lambda'}{\lambda^3}\langle \nabla^0 f, \partial_y\rangle - \langle \nabla^0 f, \nabla^0(\lambda^{-2})\rangle_0 + \langle \nu, \nabla^0(\lambda^{-2})\rangle_0 \langle \nabla^0 f, \nu\rangle\right) \\
&+ \frac{1}{\lambda^2 H^2}\left(\langle \nabla^0_{\nu}\nabla^0 f + \frac{\lambda'}{\lambda}\left(\langle \nu, \partial_y\rangle_0 \nabla^0 f + \langle \partial_y, \nabla^0 f\rangle_0 \nu - \langle \nu, \nabla^0 f\rangle_0 \partial_y\right), \nu\rangle\right) + \frac{2}{\lambda H}\nabla^0_{\nu} f \\
&= \frac{1}{\lambda^2 H^2}\left(\langle \nabla^0_{\nu}\nabla^0 f, \nu\rangle - \Delta^0 f - (n-2)\frac{\lambda'}{\lambda}\langle \nabla^0 f, \partial_y\rangle - 2\frac{\lambda'}{\lambda}\langle \nabla^0 f, \nu\rangle\langle \nu, \partial_y\rangle\right) + \frac{2}{\lambda H}\nabla^0_{\nu} f
\end{aligned}$$

$\square$

Note: g depends on t through the embedding function φ_t but if it also independently depends on t then there will be another term in the evolution equation for g corresponding to the partial derivative w.r.t this aformentioned dependance on t

Note: From now on we will be sloppy and just denote g, the function defined on Σ_t, and f, the extrinsically defined function on N, as the same function where the composition with the embedding function, φ, is implied.

Corollary 2. *If $N = \mathbb{R}^{n+1}$ then we find*

$$(\partial_t - \frac{1}{H^2}\Delta)g = \frac{2}{H}\nabla^0_\nu f + \frac{\langle \nabla^0_\nu \nabla^0 f, \nu\rangle_0 - \Delta^0 f}{H^2}$$

and if $N = \mathbb{H}^{n+1}$ then we find

$$(\partial_t - \frac{1}{H^2}\Delta)g = \frac{1}{H^2}\left(y^2\langle \nabla^0_\nu \nabla^0 f, \nu\rangle - y^2\Delta^0 f + (n-2)y\langle \nabla^0 f, \partial_y\rangle + 2y\langle \nabla^0 f, \nu\rangle\langle \nu, \partial_y\rangle\right) + \frac{2y}{H}\nabla^0_\nu f$$

Now we make the following defintion which we will use throughout the rest of the document.

Definition 1. *Let Σ_0 be a hypersurface satisfying the conditions of Theorem 1 and let Σ_t be the corresponding solution of IMCF which is guaranteed to exist for a short time and we let T be the maximal existence time. Then for $T_\epsilon < T$ we let*

$$\Omega_{R,T_\epsilon} := \overline{B}_R(0) \times [0, T_\epsilon)$$

and then we also define

$$\mathcal{H}^R_0 = \inf_{\overline{\Omega}_{R,T_\epsilon}} \min(H, H^2) > 0$$

If we consider a function $\alpha(x_1, ..., x_n, y, t, R, \mathcal{H}^R_0)$ then we can also define

$$U_R = \{(x,t) \in \Omega_{R,T} : \alpha(\varphi(x,t), t) > 0\}$$

Note: We will be interested in α such that $\overline{U}_R$ is compact and we will also want to impose the condition that $U_R \subset\subset \Omega_{R,T_\epsilon}$ so that we can use the fact that $\alpha \equiv 0$ on ∂U_R to exclude the possibility that the maximum of a positive function αg can occur on ∂U_R . We will impose this condition below through the constant C_R.

Lemma 1. *If we define $\alpha = R^2 - |x|^2 - \frac{2}{\mathcal{H}^R_0}(ny_0^2 + 4y_0 R + C_R)t$ for $N = \mathbb{H}^{n+1}$, where $y(x,0) \leq y_0$, $C_R \geq 0$ then α is a subsolution to the IMCF heat operator on Σ_t, i.e.*

$$\left(\partial_t - \frac{1}{H^2}\Delta\right)\alpha \leq -\frac{2C_R}{\mathcal{H}^R_0} \leq 0$$

Proof. If we let $|x|^2 = x_1^2 + ... + x_n^2$ for $N = \mathbb{H}^{n+1}$ then we can find

$$(\partial_t - \frac{1}{H^2}\Delta)|x|^2 = \frac{1}{H^2}\left(y^2(2|\hat{\nu}|^2 - 2n) + 4y\langle x,\nu\rangle_0\langle\nu,\partial y\rangle_0\right) + \frac{4y}{H}\langle x,\nu\rangle_0$$

where we have used the following relations as well as Corollary 1

$$\nabla^0|x|^2 = 2x \quad \nabla^0_\nu\nabla^0|x|^2 = 2\hat{\nu} \quad \Delta^0|x|^2 = 2n$$

where $\hat{\nu}$is the projection of ν onto $\mathbb{R}^n \times \{0\}$.

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)\alpha &= \frac{-1}{H^2}\left(y^2(2|\hat{\nu}|^2 - 2n) + 4y\langle x,\nu\rangle_0\langle\nu,\partial y\rangle_0\right) - \frac{4y}{H}\langle x,\nu\rangle_0 - \frac{2}{\mathcal{H}_0^R}(ny_0^2 + 4y_0R + C_R)\\ &\le \frac{2ny^2}{H^2} + \frac{4yR}{H^2} + \frac{4yR}{H} - \frac{2}{\mathcal{H}_0^R}(ny_0^2 + 4y_0R + C_R) \le -\frac{2C_R}{\mathcal{H}_0^R}\end{aligned}$$

□

Note: We purposefully leave $C_R > 0$ undefined for now because we will choose it later depending on the estimate we are trying to achieve. We do note that in order to cause $U_R \subset\subset \Omega_{R,T_\epsilon}$ we may need to choose C_R large enough to make this true. Since this choice can be made so that so that C_R is $O(R^2)$ we will not worry about it in the future (See the proof of Corollary 3).

So now we have found good cutoff functions in the sense that they are subsolutions to the IMCF heat operator. In the next two sections we will use these cutoff functions in order to obtain important estimates for non-compact IMCF.

4. Evolution Equations and First Estimate:

In this section we will derive important geometric evolution equations as well as obtain our first local estimate for the support function. We begin by specifying the notation we will use for various geometric quantities.

In the case of $\mathbb{H}^{n+1}$ we use a formula derived earlier to notice that the vector field ∂_y satisfies the following equation in $\mathbb{H}^{n+1}$

$$\bar{\nabla}_X\partial_y = -\frac{1}{y}X \tag{3}$$

and hence if we define $\eta = -\partial_y$ then we have $\bar{\nabla}_X\eta = \frac{1}{y}X$. So if we define the support function $w(x) = \bar{g}(\eta,\bar{\nu})$ then we can find the following important evolution equations

Lemma 2. *If we let* $u = \frac{1}{wH}$ *then we find*

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)w &= \frac{|A|^2}{H^2}w \\
(\partial_t - \frac{1}{H^2}\Delta)H &= -2\frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} + \frac{n}{H} \\
(\partial_t - \frac{1}{H^2}\Delta)u &= 2\frac{g(\nabla w, \nabla u)}{H^2 w^2} - \frac{nu}{H^2} \\
(\partial_t - \frac{1}{H^2}\Delta)A_i^j &= -\frac{2}{H^3}\nabla_i H \nabla^j H + \left(\frac{|A|^2}{H^2} + \frac{n}{H}\right) A_i^j - \frac{2}{H}A_{il}A^{jl}
\end{aligned}
$$

Proof. In order to compute the laplacian of w we will strongly take advantage of the fact that η satisfies equation (3). Let $\{e_1, ..., e_n\}$ be a normal frame for Σ_0 and compute

$$
\begin{aligned}
\nabla_i w &= \bar{g}(\bar{\nabla}_i \eta, \nu) + \bar{g}(\eta, \bar{\nabla}_i \nu) \\
&= \bar{g}(e_i, \nu) + A(\eta^T, e_i) = A(\eta^T, e_i)
\end{aligned}
$$

where η^T is the projection of η onto the tangent space of Σ_t.
Then we can compute the Laplacian of w

$$
\Delta w = g^{ij}\left[(\nabla_i A)(\eta^T, e_i) + A(\nabla_i \eta^T, e_i)\right]
$$

Now in order to better understand the term $A(\nabla_i \eta^T, e_i)$ we find

$$
\begin{aligned}
\frac{1}{y}e_i = \bar{\nabla}_i \eta &= \bar{\nabla}_i(\eta^T + \bar{g}(\eta, \nu)\nu) \\
&= \nabla_i \eta^T - A(e_i, \eta^T)\nu + \bar{g}(\bar{\nabla}_i \eta, \nu)\nu + \bar{g}(\eta, \bar{\nabla}_i \nu)\nu + \bar{g}(\eta, \nu)\bar{\nabla}_i \nu \\
&= \nabla_i \eta^T + \bar{g}(\eta, \nu)\bar{\nabla}_i \nu
\end{aligned}
$$

If we rearange this we find that

$$
\nabla_i \eta^T = \frac{1}{y}e_i - w\bar{\nabla}_i \nu
$$

and so we can find

$$
A(\nabla_i \eta^T, e_i) = \frac{1}{y}H - w|A|^2
$$

By a standard calculation we know that

$$(\nabla_i A)(\eta^T, e_i) = \bar{g}(\nabla H, \eta^T) + \bar{R}c(\nu, \eta^T) = \bar{g}(\nabla H, \eta^T)$$

So now if we put all of this together we find

$$\Delta w = \frac{1}{y}H - w|A|^2 + \bar{g}(\nabla H, \eta^T)$$

Now if we compute the time derivative of f under IMCF we find

$$\begin{aligned}\frac{\partial w}{\partial t} &= \bar{g}(\frac{\partial \eta}{\partial t}, \nu) + \bar{g}(\eta, \frac{\partial \nu}{\partial t})\\ &= \frac{1}{H}\bar{g}(\bar{\nabla}_\nu \eta, \nu) + \frac{1}{H^2}\bar{g}(\eta^T, \nabla H)\\ &= \frac{1}{yH} + \frac{1}{H^2}\bar{g}(\eta^T, \nabla H)\end{aligned}$$

From these two equations the evolution equation for w follows.

The evolution equation for H is well known and so we will move on to justifying the evolution equation for u

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)u &= \frac{-1}{Hw^2}(\partial_t - \frac{1}{H^2}\Delta)w - \frac{1}{H^2 w}(\partial_t - \frac{1}{H^2}\Delta)H - 2\frac{|\nabla H|}{H^5 w} - 2\frac{g(\nabla w, \nabla H)}{H^4 w^2} - 2\frac{|\nabla w|^2}{H^3 w^3}\\ &= 2\frac{g(\nabla w, \nabla u)}{H^2 w} - \frac{n}{H^3 w}\end{aligned}$$

where we have used the fact that $\nabla u = -\frac{\nabla H}{wH^2} - \frac{\nabla w}{Hw^2}$.

The evolution equation for A_i^j follows from well know evolution equations for A_{ij} [3], [16] combined with the evolution of the inverse metric g^{ij}.

□

Now we will take these three evolution equations obtained in Lemma 4 and combine them with cutoff functions in order to obtain important estimates for non-compact IMCF.

Lemma 3. *The function w^{-1} satisfies the evolution equation*

$$(\partial_t - \frac{1}{H^2}\Delta)w^{-1} = -\frac{|A|^2}{H^2}w^{-1} - \frac{2}{w^{-1}H^2}|\nabla w^{-1}|^2$$

and if we let α be a subsolution to the IMCF heat operator defined above we find

$$(\partial_t - \frac{1}{H^2}\Delta)(\alpha w^{-1}) \leq -\frac{2}{w^{-1}H^2}g(\nabla w^{-1}, \nabla(\alpha w^{-1})) - \frac{1}{n}(\alpha w^{-1}) - 2\frac{C_R}{\mathcal{H}_0^R}w^{-1}$$

Proof. We compute using the evolution of w the following

$$(\partial_t - \frac{1}{H^2}\Delta)w^{-1} = -w^{-2}(\partial_t - \frac{1}{H^2}\Delta)w - \frac{2}{w^3H^2}|\nabla w|^2 = -\frac{|A|^2}{H^2}w^{-1} - \frac{2}{w^{-1}H^2}|\nabla w^{-1}|^2$$

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)(\alpha w^{-1}) &= w^{-1}(\partial_t - \frac{1}{H^2}\Delta)\alpha + \alpha(\partial_t - \frac{1}{H^2}\Delta)w^{-1} - 2\frac{g(\nabla\alpha, \nabla w^{-1})}{H^2} \\ &\leq -\frac{|A|^2}{H^2}(\alpha w^{-1}) - \frac{2\alpha}{w^{-1}H^2}|\nabla w^{-1}|^2 - 2\frac{g(\nabla\alpha, \nabla w^{-1})}{H^2} \\ &\leq -\frac{2w}{H^2}g\left(\nabla\left(\alpha w^{-1}\right), \nabla w^{-1}\right) - \frac{|A|^2}{H^2}\alpha w^{-1} - 2\frac{C_R}{\mathcal{H}_0^R}w^{-1}\end{aligned}$$

where we are using the fact that

$$g\left(\nabla\left(\alpha w^{-1}\right), \nabla w^{-1}\right) = \alpha|\nabla w^{-1}|^2 + w^{-1}g(\nabla\alpha, \nabla w^{-1})$$

and then the last inequality follows from the fact that $|A|^2 \geq \frac{1}{n}H^2$.

□

Now we will take Lemma 3 and turn it into an estimate for w^{-1} and hence a lower bound for w. First we must make an important defintion which will also be used throughtout the next section.

Definition 2. *Let Σ_0 be a hypersurface satisfying the conditions of Theorem 1 and let Σ_t be the corresponding solution of IMCF which is guaranteed to exist for at least a short time $t \in [0, \epsilon)$. Then we let*

$$\begin{aligned}U_{R,\theta,t} &= \{(x,t) \in U_R : |x|^2 + \frac{2}{\mathcal{H}_0^R}(ny_0^2 + 4y_0R + C_R)t \leq \theta R^2\} \\ &= \{(x,t) \in U_R : \alpha(\varphi(x,t),t) \geq (1-\theta)R^2\}\end{aligned}$$

where $\theta \in (0,1)$.

Lemma 4. *If we assume that Σ_0 is a hypersurface to which the hypotheses of Theorem 1 apply, then we find that*

$$\max_{U_{R,\theta,t}} w^{-1} \leq (1-\theta)^{-1}\left(\max_{U_{R,1,0}} w^{-1}\right)e^{-t/n}$$

Proof. From Lemma 3 we have the following evolution equation

$$(\partial_t - \frac{1}{H^2}\Delta)(\alpha w^{-1}) = -\frac{2}{w^{-1}H^2}g(\nabla w^{-1}, \nabla(\alpha w^{-1})) - \frac{|A|^2}{H^2}(\alpha w^{-1})$$

So then by a Lemma of Hamilton we find

$$\frac{d}{dt}\max_{U_{R,1,t}}(\alpha w^{-1}) \le -\frac{1}{n}\max_{U_{R,1,t}}(\alpha w^{-1})$$

and so if we integrate this equation we find

$$\max_{U_{R,1,t}}(\alpha w^{-1}) \le \max_{U_{R,1,0}}(\alpha w^{-1})e^{-t/n}$$

and then using the fact that $\alpha \ge (1-\theta)R^2$ on $U_{R,\theta,t}$ we find

$$(1-\theta)R^2 \max_{U_{R,\theta,t}} w^{-1} \le \max_{U_{R,1,0}} w^{-1}R^2e^{-t/n}$$

which yields the desired result.

□

So now we have found our first local estimate which will give us a lower bound on the support function in a parabolic ball which says that Σ_t remains a graph in the parabolic ball. In the next section we will obtain a series of local estimates, which require more work then the straight forward cutoff function techniques used above, culminating in Theorem 7 and Corollary 3 .

5. Further Local Estimates:

In this section we continue to obtain detailed local estimates for solutions of IMCF where Σ_0 is a hypersurfaces satisfying the hypotheses of Theorem 1. This section ends with the full collection of local estimates for important geometric quanities in Theorem 6 as well as the corresponding global estimates as we let $R \to \infty$ in Corollary 3. The global lower bound on H, necessary to finish the proof of Theorem 1, is also contained in Corollary 3.

Lemma 5. *We can find the following evolution equation for αu*

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\alpha u) \leq & -\frac{2}{H^2}g(\nabla(w^{-1}), \nabla(\alpha u)) - \frac{2}{\alpha H^2}g(\nabla\alpha, \nabla(\alpha u)) \\
& + \frac{2u}{H^2}g(\nabla(w^{-1}), \nabla\alpha) + \frac{2u}{\alpha H^2}|\nabla\alpha|^2 - \frac{n}{H^2}(\alpha u) - 2u\frac{C_R}{\mathcal{H}_0^R}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\alpha u) &= u(\partial_t - \frac{1}{H^2}\Delta)\alpha + \alpha(\partial_t - \frac{1}{H^2}\Delta)u - 2\frac{g(\nabla\alpha, \nabla u)}{H^2} \\
&\leq -2u\frac{C_R}{\mathcal{H}_0^R} + 2\alpha\frac{g(\nabla w, \nabla u)}{H^2 w^2} - \frac{n}{H^2}(\alpha u) - 2\frac{g(\nabla\alpha, \nabla u)}{H^2}
\end{aligned}
$$

Then using the following two relations

$$
\begin{aligned}
-\frac{2}{H^2}g(\nabla(w^{-1}), \nabla(\alpha u)) &= -\frac{2\alpha}{H^2}g(\nabla(w^{-1}), \nabla u) - \frac{2u}{H^2}g(\nabla(w^{-1}), \nabla\alpha) \\
-\frac{2}{\alpha H^2}g(\nabla\alpha, \nabla(\alpha u)) &= -\frac{2}{H^2}g(\nabla\alpha, \nabla u) - \frac{2u}{\alpha H^2}|\nabla\alpha|^2
\end{aligned}
$$

we get the desired equation.

∎

Notice that we have an α in the denominator of the $|\nabla\alpha|^2$ term which we no causes problems so we will try to handle this by looking at $\eta(\alpha)$ for $\eta : \mathbb{R} \to \mathbb{R}$ where we will impose the condition that $\eta(0) = 0$, if we can.

Lemma 6. *We can find the following evolution equation for $\eta(\alpha)u$, which we will often write as ηu*

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\eta u) \leq & -\frac{2}{H^2}g(\nabla(w^{-1}), \nabla(\eta u)) - \frac{2\eta'}{\eta H^2}g(\nabla\alpha, \nabla(\eta u)) \\
& + \frac{2u\eta'}{H^2}g(\nabla(w^{-1}), \nabla\alpha) + \frac{u|\nabla\alpha|^2}{H^2}\left(\frac{2\eta'^2}{\eta} - \eta''\right) - \frac{n}{H^2}(\eta u) - 2u\eta'\frac{C_R}{\mathcal{H}_0^R}
\end{aligned}
$$

Proof. This follows by noting the following formula

$$
(\partial_t - \frac{1}{H^2}\Delta)(\eta u) = u(\partial_t - \frac{1}{H^2}\Delta)\eta + \eta(\partial_t - \frac{1}{H^2}\Delta)u - 2\eta'\frac{g(\nabla\alpha, \nabla u)}{H^2}
$$

and this formula

$$
(\partial_t - \frac{1}{H^2}\Delta)\eta \leq -\frac{2C_R\eta'}{\mathcal{H}_0^R} - \frac{\eta''|\nabla\alpha|^2}{H^2}
$$

and then following the same steps as in the proof of Lemma 6. ∎

We will not be able to impose the condition that $\eta(0) = 0$ but we will do the next best thing $\eta(0) = \epsilon$ and then we will leverage Theorem 2 in order to get an estimate for u on $U_{R,1,t}$.

Lemma 7. *Assume that Σ_0 is a hypersurface to which Theorem 1 applies then we find that*

$$\max_{U_{R,1,t}} u \leq \left(\max_{U_{R,1,0}} u \right) e^{-t/n}$$

Proof. If we consider $\eta(s) = \epsilon e^{\epsilon s}$, $0 < \epsilon < 1$ and use Lemma 6 we find that

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)(\eta u) &\leq -\frac{2}{H^2}g(\nabla(w^{-1}), \nabla(\eta u)) - \frac{2\eta'^2}{\alpha H^2}g(\nabla\alpha, \nabla(\eta u)) \\ &\quad + \frac{u}{H^2}\left(2\eta'|\nabla(w^{-1})||\nabla\alpha| + |\nabla\alpha|^2\eta''\right) - 2u\eta'\frac{C_R}{\mathcal{H}_0^R} - \frac{n}{H^2}(\eta u)\end{aligned}$$

using the fact that $\eta' = \epsilon^2 e^{\epsilon s}$ and $\eta'' = \epsilon^3 e^{\epsilon s}$

$$\frac{2\eta'^2}{\eta} - \eta'' = \frac{2(\epsilon^2 e^{\epsilon s})^2}{\epsilon e^{\epsilon s}} - \epsilon^3 e^{\epsilon s} = \epsilon^3 e^{\epsilon s} = \eta''$$

now if we use the fact that Σ_t is well defined in U_R then we know that there exists some $D_R > 0$ so that $D_R^{-2}\delta \leq g \leq D_R^2\delta$, in U_R, and hence $|\nabla\alpha| \leq D_R|\nabla^0\alpha| \leq 2D_R|x| \leq 2D_R R$. We also know that $|\nabla w^{-1}| \leq D_R'$ in U_R, which is equivalent to having a bound on $|A|^2$ further explored in Lemma 10, and so we can rewrite the equation as

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)(\eta u) &\leq -\frac{2}{H^2}g(\nabla(w^{-1}), \nabla(\eta u)) - \frac{2\eta'^2}{\alpha H^2}g(\nabla\alpha, \nabla(\eta u)) \\ &\quad + \frac{u}{\mathcal{H}_0^R}\left(4D_R D_R' R\eta' + 4D_R R^2\eta'' - 2\eta' C_R\right) - \frac{n}{H^2}(\eta u)\end{aligned}$$

So now if we use the fact that $\eta \geq \eta' \geq \eta''$ then we can choose $C_R \geq 4D_R D_R' R + 4D_R R^2$ then we find that

$$(\partial_t - \frac{1}{H^2}\Delta)(\eta u) \leq -\frac{2}{H^2}g(\nabla(w^{-1}), \nabla(\eta u)) - \frac{2\eta'^2}{\alpha H^2}g(\nabla\alpha, \nabla(\eta u)) - \frac{n}{H^2}(\eta u)$$

Now if we know that $H \leq H_1$ on U_R and then define $\Phi_\delta(x,t) = c_0 e^{\frac{-n}{H_1^2}t} - \eta u + \delta t$ where $c_0 = \max_{U_{R,1,0}} \eta u > 0$ then we find

$$(\partial_t - \frac{1}{H^2}\Delta)\Phi_\delta \geq -\frac{2}{H^2}g(\nabla(w^{-1}), \nabla(\eta u)) - \frac{2\eta'^2}{\alpha H^2}g(\nabla\alpha, \nabla(\eta u)) - \frac{n}{H_1^2}\Phi_0 + \delta$$

Now let $(x_0, t_0) \in U_R$, for sake of contradiction, be such that $\Phi_\delta(x_0, t_0) = \min_{\bar{U}_R} \Phi_\delta(x,t) < 0$ and hence $\nabla\Phi_\delta(x_0, t_0) = 0$, $\Delta\Phi_\delta(x_0, t_0) \geq 0$ and $\frac{\partial\Phi_\delta}{\partial t}(x_0, t_0) = \frac{\partial\Phi_0}{\partial t}(x_0, t_0) + \delta \leq 0$

So by the evolution equation above we find that $(\partial_t - \frac{1}{H^2}\Delta)\Phi_\delta(x_0, t_0) > 0$ but on the contrary by our deductions at (x_0, t_0) we find that $(\partial_t - \frac{1}{H^2}\Delta)\Phi_\delta(x_0, t_0) \leq 0 \longrightarrow\!\longleftarrow$ hence by letting $\delta \to 0$ we find that Φ_0 cannot obtain a negative minimum on U_R which implies that a negative minimum of Φ_0 can only be obtained on the set $\partial_p U_R = U_{R,1,0} \cup \{\alpha = 0\}$.

If the min occurs on $\{\alpha = 0\}$ then we have that $\eta(\alpha) = \epsilon$ on this set and hence $c_0 e^{\frac{-n}{H_1^2}t} - \epsilon u < 0$ on $\{\alpha = 0\}$. Since we know that u is bounded on $\bar{U}_R$ we can choose $\epsilon < c_0 \left(\max_{\{\alpha=0\}} u\right)^{-1} e^{\frac{-n}{H_1^2}t} = \epsilon_0$ and take note of the fact that $T < \infty$ for this estimate and hence find that $c_0 e^{\frac{-n}{H_1^2}t} - \epsilon u > 0 \longrightarrow\!\longleftarrow$ and so we know that Φ_0 cannot obtain a negative min on $\{\alpha = 0\}$ for $\epsilon \in (0, \epsilon_0)$ where $\epsilon_0 > 0$.

It is clear by the construction of Φ_0 that it cannot obtain a negative min on $U_{R,1,0}$ and hence $\Phi_0(x,t) \geq 0$ which means that

$$\max_{U_{R,1,t}} \eta u \leq \left(\max_{U_{R,1,0}} \eta u\right) e^{\frac{-n}{H_1^2}t}$$

Since we know that $\alpha \geq (1-\theta)R^2$ on $U_{R,\theta,t}$ for $\theta \in (0,1)$ we know that $\eta(\alpha) \geq \epsilon e^{\epsilon(1-\theta)R^2}$ and so we have

$$\epsilon e^{\epsilon(1-\theta)R^2} \max_{U_{R,\theta,t}} u \leq \left(\max_{U_{R,1,0}} u\right) e^{\frac{-n}{H_1^2}t} \epsilon e^{\epsilon R^2} \quad \Rightarrow \quad \max_{U_{R,\theta,t}} u \leq \left(\max_{U_{R,1,0}} u\right) e^{\frac{-n}{H_1^2}t} e^{\epsilon\theta R^2}$$

and so by letting $\epsilon \to 0$ we find that

$$\max_{U_{R,\theta,t}} u \leq \left(\max_{U_{R,1,0}} u\right) e^{\frac{-n}{H_1^2}t}$$

and then by noticing that the estimate does not depend on θ we can let $\theta \to 1$ which gives us the desired result.

$\square$

Now we obtain a upper bound on H in $U_{R,\theta,t}$.

Lemma 8. *If we assume that Σ_0 is a hypersurface to which Theorem 1 applies then we can find*

$$\max_{U_{R,\theta,t}} H \leq (1-\theta)^{-1}\left(\max_{U_{R,1,0}} H\right) e^{n\left(\frac{1}{H_{0,R}^2}-\frac{1}{n^2}\right)t}$$

Proof. For this we will look at αH which has a nice evolution equation so we can choose $C_R = 0$ and find

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)(\alpha H) &= H(\partial_t - \frac{1}{H^2}\Delta)\alpha + \alpha(\partial_t - \frac{1}{H^2}\Delta)H - 2\frac{g(\nabla\alpha, \nabla H)}{H^2} \\ &\leq -2\alpha\frac{|\nabla H|^2}{H^3} - 2\frac{g(\nabla\alpha,\nabla H)}{H^2} + \left(\frac{n}{H^2} - \frac{|A|^2}{H^2}\right)(\alpha H) \\ &\leq -2\frac{g(\nabla(\alpha H), \nabla H)}{H^3} + n\left(\frac{1}{H_{0,R}^2} - \frac{1}{n^2}\right)(\alpha H)\end{aligned}$$

Since we know that H is bounded from below in U_R we can proceed in the following fassion

$$\frac{d}{dt}\max_{U_{R,1,t}}(\alpha H) \leq n\left(\frac{1}{H_{0,R}^2} - \frac{1}{n^2}\right)\max_{U_{R,1,t}}(\alpha H)$$

and so if we integrate this equation we find

$$\max_{U_{R,1,t}}(\alpha H) \leq \max_{U_{R,1,0}}(\alpha H) e^{n\left(\frac{1}{H_{0,R}^2}-\frac{1}{n^2}\right)t}$$

and then using the fact that $\alpha \geq (1-\theta)R^2$ on $U_{R,\theta,t}$ we find

$$(1-\theta)R^2 \max_{U_{R,\theta,t}} H \leq \left(\max_{U_{R,1,0}} H\right) R^2 e^{n\left(\frac{1}{H_{0,R}^2}-\frac{1}{n^2}\right)t}$$

which yields the desired result. $\square$

The last local estimate we will obtain is a second order estimate for the graph function y and r which we will obtain through bounding A_{ij}. We start by obtaining important evolution equations and then obtain the estimate in Lemma 10.

Lemma 9. *If we define $P_i^j = w^{-1}A_i^j$ then we will find the following evolution equation*

$$(\partial_t - \frac{1}{H^2}\Delta)P_i^j = -\frac{2}{wH^3}\nabla_i H\nabla^j H - \frac{2w}{H^2}g(\nabla w^{-1}, \nabla P_i^j) + \frac{n}{H}P_i^j - \frac{2w}{H}(P^2)_i^j$$

Now if we consider $\eta : \mathbb{R} \to \mathbb{R}$ and α the cutoff function from Lemma 1 or 2 then we find the following evolution equation for $\eta(\alpha)P_i^j$

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)(\eta P_i^j) = &-\frac{2w}{H^2}g(\nabla w^{-1}, \nabla(\eta P_i^j)) - \frac{2}{\alpha H^2}g(\nabla\eta, \nabla(\eta P_i^j))\\ &- \frac{2\eta}{wH^3}\nabla_i H\nabla^j H + \frac{2w\eta'}{H^2}P_i^j g(\nabla w^{-1}, \nabla\alpha) + \frac{2P_i^j}{H^2}\frac{\eta'^2}{\eta}|\nabla\alpha|^2 - \frac{\eta'' P_i^j}{H^2}|\nabla\alpha|^2\\ &+ \frac{n}{H}(\eta P_i^j) - \frac{2w}{\eta H}(\eta^2 P^2)_i^j - \frac{2C_R\eta' P_i^j}{\mathcal{H}_0^R}\end{aligned}$$

Proof.

$$\begin{aligned}(\partial_t - \frac{1}{H^2}\Delta)P_i^j &= w^{-1}(\partial_t - \frac{1}{H^2}\Delta)A_i^j + A_i^j(\partial_t - \frac{1}{H^2}\Delta)w^{-1} - \frac{2}{H^2}g(\nabla w^{-1}, \nabla A_i^j)\\ &= w^{-1}\left(-\frac{2}{H^3}\nabla_i H\nabla^j H + \frac{|A|^2}{H^2}A_i^j + \frac{n}{H}A_i^j - \frac{2}{H}(A^2)_i^j\right)\\ &\quad - \frac{|A|^2}{H^2}w^{-1}A_i^j - \frac{2wA_i^j}{H^2}|\nabla w^{-1}|^2 - \frac{2}{H^2}g(\nabla w^{-1}, \nabla A_i^j)\\ &= -\frac{2}{wH^3}\nabla_i H\nabla^j H - \frac{2wA_i^j}{H^2}|\nabla w^{-1}|^2 - \frac{2}{H^2}g(\nabla w^{-1}, \nabla A_i^j) + \frac{n}{H}P_i^j - \frac{2w}{H}(P^2)_i^j\end{aligned}$$

Now we will use the fact that

$$-\frac{2w}{H^2}g(\nabla w^{-1}, \nabla(w^{-1}A_i^j)) = -\frac{2wA_i^j}{H^2}|\nabla w^{-1}|^2 - \frac{2}{H^2}g(\nabla w^{-1}, \nabla A_i^j)$$

to find the following

$$(\partial_t - \frac{1}{H^2}\Delta)P_i^j = -\frac{2}{wH^3}\nabla_i H\nabla^j H - \frac{2w}{H^2}g(\nabla w^{-1}, \nabla P_i^j) + \frac{n}{H}P_i^j - \frac{2w}{H}(P^2)_i^j$$

Note: We are not worried about the $\nabla_i H\nabla^j H$ term since at some point in this argument we are going to look at the maximum eigenvalue of P_i^j in which case this term will be negative.

Now if we let α be the cutoff function from Lemma 2 so that $(\partial_t - \frac{1}{H^2}\Delta)\alpha \leq -\frac{2C_R}{\mathcal{H}_0^R}$ then we can compute the following evolution equation for αP_i^j

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\alpha P_i^j) &= \alpha(\partial_t - \frac{1}{H^2}\Delta)P_i^j + P_i^j(\partial_t - \frac{1}{H^2}\Delta)\alpha - \frac{2}{H^2}g(\nabla\alpha, \nabla P_i^j) \\
&= -\frac{2\alpha}{wH^3}\nabla_i H\nabla^j H - \frac{2w\alpha}{H^2}g(\nabla w^{-1}, \nabla P_i^j) - \frac{2}{H^2}g(\nabla\alpha, \nabla P_i^j) \\
&+ \frac{n\alpha}{H}P_i^j - \frac{2w\alpha}{H}(P^2)_i^j - \frac{2C_R P_i^j}{\mathcal{H}_0^R}
\end{aligned}
$$

Now we again compute some gradient terms

$$
\begin{aligned}
-\frac{2w}{H^2}g(\nabla w^{-1}, \nabla(\alpha P_i^j)) &= -\frac{2\alpha w}{H^2}g(\nabla w^{-1}, \nabla P_i^j) - \frac{2w}{H^2}P_i^j g(\nabla w^{-1}\nabla\alpha) \\
-\frac{2}{\alpha H^2}g(\nabla\alpha, \nabla(\alpha P_i^j)) &= -\frac{2}{H^2}g(\nabla\alpha, \nabla P_i^j) - \frac{2P_i^j}{\alpha H^2}|\nabla\alpha|^2
\end{aligned}
$$

from which we find

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\alpha P_i^j) &= -\frac{2w}{H^2}g(\nabla w^{-1}, \nabla(\alpha P_i^j)) - \frac{2}{\alpha H^2}g(\nabla\alpha, \nabla(\alpha P_i^j)) \\
&- \frac{2\alpha}{wH^3}\nabla_i H\nabla^j H + \frac{2w}{H^2}P_i^j g(\nabla w^{-1}, \nabla\alpha) + \frac{2P_i^j}{\alpha H^2}|\nabla\alpha|^2 \\
&+ \frac{n}{H}(\alpha P_i^j) - \frac{2w}{\alpha H}(\alpha^2 P^2)_i^j - \frac{2C_R P_i^j}{\mathcal{H}_0^R}
\end{aligned}
$$

It may be possible to make the argument here but we don't like the α that shows up in the denominator of the $|\nabla\alpha|^2$ term and so to make the argument cleaner we consider a function $\eta : \mathbb{R} \to \mathbb{R}$ and compute the following evolution for $\eta(\alpha)P_i^j$

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\eta P_i^j) &= -\frac{2w}{H^2}g(\nabla w^{-1}, \nabla(\eta P_i^j)) - \frac{2}{\alpha H^2}g(\nabla\eta, \nabla(\eta P_i^j)) \\
&- \frac{2\eta}{wH^3}\nabla_i H\nabla^j H + \frac{2w\eta'}{H^2}P_i^j g(\nabla w^{-1}, \nabla\alpha) + \frac{2P_i^j}{H^2}\frac{\eta'^2}{\eta}|\nabla\alpha|^2 - \frac{\eta'' P_i^j}{H^2}|\nabla\alpha|^2 \\
&+ \frac{n}{H}(\eta P_i^j) - \frac{2w}{\eta H}(\eta^2 P^2)_i^j - \frac{2C_R\eta' P_i^j}{\mathcal{H}_0^R}
\end{aligned}
$$

$\square$

Now we are ready to prove an estimate for P_i^j which will imply an estimate for A_i^j.

Lemma 10. *If we define $P_i^j = w^{-1}A_i^j$ and we assume that Σ_0 is a hypersurface to which Theorem 1 applies then we find that*

$$\max_{U_{R,1,t}} P_i^j \leq \max\left(\max_{U_{R,1,0}} P_i^j, \frac{nc_0}{2}\right)$$

where $\max\limits_U P_i^j$ refers to the maximum eigenvalue of P over the set U.

Proof. Now we would like to better understand some terms in this equation and get rid of the bad terms

$$\frac{2w\eta'}{H^2}g(\nabla w^{-1},\nabla\alpha) = \frac{-2w^{-1}\eta'}{H^2}g(\nabla w,\nabla\alpha) \leq \frac{2w^{-1}\eta'}{H^2}|\nabla w||\nabla\alpha|$$

where we have used Youngs inequality and now we notice that as before the metric g of Σ_t is unifromly equivalent to δ in U_R and hence $|\nabla\alpha| \leq C|\nabla^0\alpha| \leq 2CR$. Now we note that for a vector v tangent to Σ_t we have

$$\nabla_v w = \nabla_v \bar{g}(\nu,\eta) = \bar{g}(\bar{\nabla}_v\nu,\eta^T) = A(v,\eta^T) \quad \Rightarrow \quad |\nabla w|^2 \leq |A|^2$$

which we will use shortly. Now if we choose $\eta(s) = \epsilon e^{\epsilon s}$ then we know that $\frac{2\eta'^2}{\eta} - \eta'' = \eta''$ and so we can find

$$\begin{aligned}\frac{2\eta' w^{-1}}{H^2}|A||\nabla\alpha| + \frac{|\nabla\alpha|^2}{H^2}\left(\frac{2\eta'^2}{\eta} - \eta''\right) - \frac{2C_R\eta'}{\mathcal{H}_0^R} &\leq \frac{1}{\mathcal{H}_0^R}\left(2\eta' w^{-1}C_0R + \eta''R^2 - 2C_R\eta'\right)\\ &\leq \frac{2\eta'}{\mathcal{H}_0^R}\left(CR^2 - C_R\right)\end{aligned}$$

where we are using the fact that w^{-1} is bounded and we are assuming an upper bound on $|A|$ in the set U_R which is justified by Theorem 2. Now we can choose $C_R \geq CR^2$ in order to get rid of the bad gradient terms that come from the cutoff function.

Now we look to understanding the zero order terms $\frac{n}{H}(\eta P_i^j) - \frac{2w}{\eta H}(\eta^2P^2)_i^j$. Now if we let λ be the largest eigenvalue of P_i^j at a point $(x,t) \in U_R$, then we find the following

$$-\frac{2w}{\eta H}(\eta\lambda)^2 + \frac{n}{H}(\eta\lambda) = -\frac{2w}{H}\lambda\left(\eta\lambda - \frac{1}{2}n\eta w^{-1}\right) \leq -\frac{2w}{H}\lambda\left(\eta\lambda - \frac{1}{2}nc_0\eta\right)$$

at the point (x,t) where c_0 is a bound on w^{-1}, guaranteed by a previous estimate.

Now we are ready to give the proof of the desired result. Let $\Phi_i^j = C\delta_i^j - \eta P_i^j$ where $C = \max\left(C_0, \frac{nc_0\epsilon}{2}e^{\epsilon R^2}\right)$ and C_0 is the maximum eigenvalue of ηP_i^j in the set $U_{R,1,0}$. The goal is to show that the minimum eigenvalue of Φ_i^j is positive.

For sake of contradiction assume that the minimum eignenvalue over U_R of Φ_i^j, call it β , is strictly negative and occurs for the first time at the point $(x_0, t_0) \in U_R$ with eigenvector $v \in T_{x_0}\Sigma_{t_0}$. Then let $\lambda = P_i^j v^i v_j$ and if we use parralel translation to extend v to be constant w.r.t. Σ_t in a neighborhood of (x_0, t_0) then we find the following evolution inequality at the point (x_0, t_0)

$$
\begin{aligned}
(\partial_t - \frac{1}{H^2}\Delta)(\eta P_i^j v^i v_j) &= -\frac{2w}{H^2} g(\nabla w^{-1}, \nabla(\eta P_i^j v^i v_j)) - \frac{2}{\alpha H^2} g(\nabla \eta, \nabla(\eta P_i^j v^i v_j)) \\
&\quad - \frac{2\eta}{wH^3}\nabla_v H \nabla^v H + \frac{2w\eta'}{H^2} P_i^j v^i v_j g(\nabla w^{-1}, \nabla\alpha) + \frac{2P_i^j v^i v_j}{H^2}\left(\frac{\eta'^2}{\eta} - \eta''\right)|\nabla\alpha|^2 \\
&\quad + \frac{n}{H}(\eta P_i^j v^i v_j) - \frac{2w}{\eta H}(\eta^2 P^2)_i^j v^i v_j - \frac{2C_R \eta' P_i^j v^i v_j}{\mathcal{H}_0^R} \\
&\leq \frac{n}{H}(\eta P_i^j v^i v_j) - \frac{2w}{\eta H}(\eta^2 P^2)_i^j v^i v_j
\end{aligned}
$$

which yields the following inequality for β at the point (x_0, t_0) where we choose C_R as above

$$
(\partial_t - \frac{1}{H^2}\Delta)\beta \geq \frac{2w}{H}\lambda\left(\eta\lambda - \frac{1}{2}nc_0\eta\right) > 0
$$

where the inequality follows since $\beta = C - \lambda\eta < 0$ so $\lambda\eta > C$ and C was chosen to be larger than $\frac{1}{2}nc_0\eta$.

By our assumptions though we know that $\frac{\partial\beta}{\partial t} \leq 0$ and $\Delta\beta \geq 0$ and hence we find

$$
(\partial_t - \frac{1}{H^2}\Delta)\beta \leq 0
$$

which is a contradiction so Φ_i^j cannot obtain a strictly negative minimum eigenvalue on U_R.

Now we want to show that Φ_i^j cannot obtain a strictly negative minimum eigenvalue on $\{\alpha = 0\}$. If it did then we know that $\eta = \epsilon$ on $\{\alpha = 0\}$ and so we would have that $\beta = C - \epsilon\lambda < 0$. Since we can assume that $P_i^j \leq C_1\delta_i^j$ over the set U_R by a short time existence result we can choose $\epsilon < \frac{C}{C_1}$ and then $\beta \geq 0$ and hence this cannot happen.

Then we see by construction that Φ_i^j does not obtain a negative eigenvalue at time $t = 0$ and so it doesn't obtain one anywhere on U_R and hence ηP_i^j is bounded from above, as desired.

More specifically we have that

$$\max_{U_{R,1,t}} \eta P_i^j \leq C = \max\left(\max_{U_{R,1,0}} \eta P_i^j, \frac{nc_0}{2}\epsilon e^{\epsilon R^2}\right)$$

Since $\alpha \geq (1-\theta)R^2$ on $U_{R,\theta,t}$ we have that

$$\epsilon e^{\epsilon(1-\theta)R^2} \max_{U_{R,\theta,t}} P_i^j \leq C = \max\left(\max_{U_{R,1,0}} P_i^j, \frac{nc_0}{2}\right)\epsilon e^{\epsilon R^2}$$

and so we find

$$\max_{U_{R,\theta,t}} P_i^j \leq C = \max\left(\max_{U_{R,1,0}} P_i^j, \frac{nc_0}{2}\right) e^{\epsilon\theta R^2}$$

Then by letting $\epsilon \to 0$ and then letting $\theta \to 1$ we find the desired result.

□

Now we unpack all of these lemmas to see what they say about important geometric quanities.

Theorem 6. *If we assume that Σ_0 is a hypersurface to which the hypotheses of Theorem 1 apply then we find the following estimates*

$$\begin{aligned}
v(x,t) &\leq (1-\theta)^{-1}\left(\max_{U_{R,1,0}} w^{-1}\right)\left(\min_{\Sigma_0} y\right)^{-1} \\
H(x,t) &\geq y_0\left(\min_{U_{R,1,0}} H\right)\left(\min_{U_{R,1,0}} w\right) \\
H(x,t) &\leq (1-\theta)^{-1}\left(\max_{U_{R,1,0}} H\right) e^{n\left(\frac{1}{H_{0,R}^2}-\frac{1}{n^2}\right)t} \\
A_i^j(x,t) &\leq \max\left(\max_{U_{R,1,0}} A_i^j, \frac{n}{2}\right)\left(\max_{U_{R,1,0}} w^{-1}\right)\left(\min_{\Sigma_0} y\right)^{-1} e^{t/n}
\end{aligned}$$

where $y_0 = \min_{\Sigma_0} y$ and all of the estimates are valid on $U_{R,\theta,t}$.

Proof. This corollary follows from combining and unpacking Lemmas 4,7,8,10,11 as follows.

In this case $w^{-1} = yv$ and hence Lemma 4 tells us that

$$v \leq (1-\theta)^{-1} y^{-1} \left(\max_{U_{R,1,0}} w^{-1} \right) e^{-t/n}$$

and hence if we combine this estimate with Theorem 3 we get the first result.

Lemma 7 tells us that

$$\frac{w^{-1}}{\max_{U_{R,1,0}} u} e^{t/n} \leq H$$

and so if we notice that $w \leq \frac{1}{y}$ and combine with Theorem 3 we find

$$H(x,t) \geq \frac{\min_{U_{R,\theta,t}} y}{\max_{U_{R,1,0}} u} e^{t/n} = \frac{\min_{U_{R,\theta,t}} y}{\max_{U_{R,1,0}} \frac{1}{Hw}} e^{t/n} \geq y_0 \left(\min_{U_{R,1,0}} H \right) \left(\min_{U_{R,1,0}} w \right)$$

The third estimate follows directly from Lemma 8 and the last estimate follows from combining Theorem 3 with Lemma 10.

□

In the next Corollary to Theorem 6 we show that all the local estimates obtained in the last section are uniformly controlled as $R \to \infty$ and hence extend to estimates on all of Σ_t.

Corollary 3. *If we assume that Σ_0 is a hypersurface to which the hypotheses of Theorem 1 apply then all the estimates of Corollary 2 remain true over the set $\overline{U} = \bigcup_{R>0} U_R$ which is non-degenerate, i.e. $\mathbb{R}^n \times [0,\epsilon) \subset \overline{U}$ for some $\epsilon > 0$.*

$$\begin{aligned}
v(x,t) &\leq (1-\theta)^{-1} \left(\max_{\Sigma_0} w^{-1} \right) \left(\min_{\Sigma_0} y \right)^{-1} \\
H(x,t) &\geq \left(\min_{\Sigma_0} y \right) \left(\min_{\Sigma_0} H \right) \left(\min_{\Sigma_0} w \right) \\
H(x,t) &\leq (1-\theta)^{-1} \left(\max_{\Sigma_0} H \right) e^{n\left(\frac{1}{H_{0,R}^2} - \frac{1}{n^2} \right) t} \\
A_i^j &\leq \max \left(\max_{\Sigma_0} A_i^j, \frac{n}{2} \right) \left(\max_{\Sigma_0} w^{-1} \right) \left(\min_{\Sigma_0} y \right)^{-1}
\end{aligned}$$

Proof. As we attempt to extend the estimates we noted in Theorem 6 we notice that by Theorem 2 we do not worry about the assumptions on bounds on various geometric quanities since we already have global bounds for these quantities that exist on $\mathbb{R}^n \times [0, T_\epsilon)$. The one issue we need to resolve is that if C_R grows to quickly then it is possible for U to be degenerate, i.e. $\Sigma_0 \times [0, \epsilon) \not\subset U$ for any $\epsilon > 0$, as $R \to \infty$.

First, we note that since $\mathcal{H}_0^R$ is uniformly bounded in R, then at worst we can get away with $C_R = O(R^2)$ which we can see by assuming that $C_R = C_0 + C_1 R + C_2 R^2$ and noticing

$$\alpha > 0 \quad \Rightarrow \quad R^2 - |x|^2 - \frac{2}{\mathcal{H}_0^R}\left(ny_0^2 + 4y_0 R + C_R\right) t > 0 \quad \Rightarrow \quad t < \frac{\mathcal{H}_0^R (R^2 - |x|^2)}{2(ny_0^2 + 4y_0 R + C_0 + C_1 R + C_2 R^2)}$$

which for fixed x has a limit as $R \to \infty$ and tells us that $t < \frac{\mathcal{H}_0^\infty}{C_2}$ and hence U is non-degenerate , as desired. One can go back and check that any time we chose C_R in the above lemmas it was always $O(R^2)$, as required. $\square$

6. Appendix: ODE Maximum Principle at Infinity

In this section we state and prove an ODE maximum principle that works for functions defined on non-compact domains and has been used a few times in this document for important estimates. This is an extension of the work of Hamilton [12] which is described in detail in [18].

Theorem 7. *Assume for $t \in (0, T)$ that $g(t)$ is a family of Riemannian metric defined on the manifold M^n so that the dependence on t is smooth. We also assume that g_t is a metric to which the Omori-Yau maximum principle at infinity applies for each $t \in (0, T)$.*

Let $u : M \times [0, T) \to \mathbb{R}$ be a smooth function which is bounded for each time $t \in (0, T)$, i.e. $|u(x, t)| \le C(t)$, satisfying

$$\left(\partial_t - H^{ij} \nabla_i^{g_t} \nabla_j^{g_t}\right) u = \langle X(x, u, \nabla^{g_t} u, t), \nabla^{g_t} u \rangle_{g_t} + F(u)$$

where $|X| \le C'(t)$, F is a locally Lipschitz function on $\mathbb{R}$ and H_{ij} is a symmetric, non-negative matrix so that $H_{ij} \le C_0 g_{ij}$.

Setting $u_{sup}(t) = \sup_{x \in M} u(x, t)$ we have that the function is locally Lipschitz and hence differentiable at almost every time $t \in [0, T)$. At every differentiable time we have that

$$\frac{du_{sup}(t)}{dt} = \lim_{k \to \infty} \frac{\partial u(x_k, t)}{\partial t} \qquad \text{where } \{x_k\} \subset \mathbb{R}^n \text{ is a sequence so that } \lim_{k \to \infty} u(x_k, t) = \sup_{x \in \mathbb{R}^n} u(x, t)$$

and so if $\varphi : [0, T') \to \mathbb{R}$ is a maximal solution of the ODE

$$\begin{cases} \varphi'(t) & = F(\varphi(t)) \\ \varphi(0) & = u_{sup}(0) \end{cases}$$

then we have that $u(x,t) \leq \varphi(t)$ for $(x,t) \in M \times [0, T')$.

Note: We did not put a condition on g_t which implies the Omori-Yau maximum principle at infinity since there are fairly general analytic conditions which guarantee its application that may be useful in different circumstances (See [1], [20], [26] and especially [21] for the most general case). So we refrain from putting a specific requirement but we note that Ricci curvature bounded from below is a sufficient geometric condition.

Before we can prove this theorem we will need the following lemma.

Lemma 11. *Let $u : M^n \times (0,T) \to \mathbb{R}$ be a bounded C^1 function then $u_{sup} : (0,T) \to \mathbb{R}$, defined as $u_{sup}(t) = \sup_{x \in M} u(x,t)$, is a locally Lipschitz function in $(0,T)$. Also, at every differentiable time $t \in (0,T)$ we have that*

$$\frac{du_{sup}(t)}{dt} = \frac{\partial u(x,t)}{\partial t} \qquad \textit{where } x \in M \textit{ is a point where } u(\cdot, t) \textit{ attains its max}$$

or

$$\frac{du_{sup}(t)}{dt} = \lim_{k\to\infty} \frac{\partial u(x_k,t)}{\partial t} \qquad \textit{where } \{x_k\} \subset M \textit{ is a sequence so that } \lim_{k\to\infty} u(x_k,t) = \sup_{x\in M} u(x,t)$$

Proof. **Proof of Lemma 11:**

Fix a $t \in (0,T)$ and then choose a $\delta > 0$ so that $(t-\delta, t+\delta) \subset (0,T)$. Then choose an ϵ so that $0 < \epsilon < \delta$ and note that since u is bounded and C^1 on $M \times (0,T)$ we know that for every $x \in M$, there exists some Lipschitz constant $K > 0$ so that $u(x, t+\epsilon) - u(x,t) \leq K\epsilon$.

Now for each ϵ we can find a sequence $\{x_k^\epsilon\}$ so that $u_{sup}(t+\epsilon) = \lim_{k\to\infty} u(x_k^\epsilon, t+\epsilon)$ and hence

$$u_{sup}(t+\epsilon) = \lim_{k\to\infty} u(x_k^\epsilon, t+\epsilon) \leq \limsup_{k\to\infty} u(x_k^\epsilon, t) + K\epsilon \leq \lim_{k\to\infty} u(x_k^0, t) + K\epsilon = u_{sup}(t) + K\epsilon$$

and so we have found that $u_{sup}(t+\epsilon) - u_{sup}(t) \leq K\epsilon$. Repeating this argument for $-\delta < \epsilon < 0$ we conclude that u_{sup} is a locally Lipschitz function on $(0,T)$ and hence differentiable at almost every time t.

Note: If u attains its max at some point $x \in M$ then we can take the trivial sequence which is constantly equal to x.

Let $t \in (0,T)$ be a time where u_{sup} is differentiable and let $\{x_k\}$ be a sequence so that $\lim_{k\to\infty} u(x_k,t) = \sup_{x\in M} u(x,t)$. Then by the Mean Value Theorem, for every $0 < \epsilon < \delta$ we can choose a $s_k \in (t, t+\epsilon)$ so that $u(x_k, t+\epsilon) = u(x_k,t) + \epsilon\frac{\partial u(x_k,s_k)}{\partial t}$ and so

$$u_{sup}(t+\epsilon) \geq \limsup_{k\to\infty} u(x_k, t+\epsilon) = \limsup_{k\to\infty}\left[u(x_k,t) + \epsilon\frac{\partial u(x_k,s_k)}{\partial t}\right] = u_{sup}(t) + \epsilon \limsup_{k\to\infty}\frac{\partial u(x_k,s_k)}{\partial t}$$

so then by rearranging we find

$$\frac{u_{sup}(t+\epsilon) - u_{sup}(t)}{\epsilon} \geq \limsup_{k\to\infty}\frac{\partial u(x_k,s_k)}{\partial t}$$

and so by letting $\epsilon \to 0$ we find that $\dfrac{du_{sup}(t)}{dt} \geq \limsup_{k\to\infty}\dfrac{\partial u(x_k,t)}{\partial t}$.
Now if we repeat this argument for $-\delta < -\epsilon < 0$ we will get the following

$$\frac{du_{sup}(t)}{dt} \leq \liminf_{k\to\infty}\frac{\partial u(x_k,t)}{\partial t}$$

Putting this all together we see that

$$\limsup_{k\to\infty}\frac{\partial u(x_k,t)}{\partial t} \leq \frac{du_{sup}(t)}{dt} \leq \liminf_{k\to\infty}\frac{\partial u(x_k,t)}{\partial t}$$

which tells us that $\lim_{k\to\infty}\dfrac{\partial u(x_k,t)}{\partial t}$ must converge at a differentiable time of $u_{sup}(t)$ and equal its derivative.

□

Proof. **Proof of Theorem 7:**
By the previous Lemma we know that $u_{sup}(t)$ is locally Lipschitz and hence differentiable almost everywhere in $[0,T)$. If we let $t \in [0,T)$ be a differentiable time and $\{x_k\}$ a sequence so that $\lim_{k\to\infty} u(x_k,t) = \sup_{x\in M} u(x,t)$, $|\nabla u(x_k,t)| < \frac{1}{k}$ and $\nabla_i\nabla_j u(x_k,t) < \frac{1}{k}g_{ij}$, which is guaranteed by the maximum principle at infinity, then we find

$$
\begin{aligned}
\frac{du_{sup}}{dt}(t) &= \lim_{k\to\infty} \frac{\partial u}{\partial t}(x_k,t) \\
&\le \limsup_{k\to\infty} \left(H^{ij}\nabla_i\nabla_j u(x_k,t) + \langle X(x_k,u,\nabla u,t), \nabla u(x_k,t)\rangle + F(u(x_k,t))\right) \\
&\le \limsup_{k\to\infty} \left(\frac{nC_0}{k} + \frac{|X|}{k} + F(u(x_k,t))\right) \\
&\le F\left(\limsup_{k\to\infty} u(x_k,t)\right) = F(u_{sup}(t))
\end{aligned}
$$

and so we have that

$$\frac{du_{sup}}{dt}(t) \le F(u_{sup}(t))$$

Now let $\varphi : [0,T') \to \mathbb{R}$ be as in the statement of the Theorem and for $\epsilon > 0$ let $\varphi_\epsilon : [0,T_\epsilon) \to \mathbb{R}$ be the maximal solution of the family of ODEs

$$\begin{cases} \varphi_\epsilon'(t) & = F(\varphi_\epsilon(t)) \\ \varphi_\epsilon(0) & = u_{sup}(0) + \epsilon \end{cases}$$

First we notice that for $\epsilon_1 \le \epsilon_2$ we have that $\varphi_{\epsilon_1}(t) \le \varphi_{\epsilon_2}$ (by ODE comparsion) and hence $T_{\epsilon_2} \le T_{\epsilon_1}$. So T_ϵ is an increasing sequence of times as $\epsilon \to 0$ and so $T_\epsilon \to T_0$ as $\epsilon \to 0$ and then by continuity of solutions on the intital condition we have that $T_0 = T'$.

Since F is Lipschitz on compact sets we can restrict ourselves to $[0,T_\delta]$ for $T_\delta < T'$ where we know that u and φ_ϵ are bounded, for small enough ϵ, and hence solutions to the above ODE have continuous dependence on the initial conditions. Hence using the fact that the family of functions φ_ϵ is uniformly Lipschitz for small enough ϵ we find that $\varphi_\epsilon \to \varphi$ uniformly on $[0,T_\delta]$ for any $T_\delta < T'$ as $\epsilon \to 0$.

Now for sake of contradiction assume that there is some positive time so that $u_{sup}(t) > \varphi_\epsilon(t)$ and let $\bar{t} > 0$ be the infimum of all such times which we know is $\neq 0$ since $u_{sup}(0) = \varphi_\epsilon(0) - \epsilon$. So $u_{sup}(\bar{t}) = \varphi_\epsilon(\bar{t})$ and hence we can let $\Phi_\epsilon(t) = \varphi_\epsilon(t) - u_{sup}(t)$. Then at differentiable times for $u_{sup}(t)$ in the interval $[0,\bar{t})$ we know that $\Phi_\epsilon(t) > 0$ and

$$\Phi_\epsilon'(t) \ge F(\varphi_\epsilon(t)) - F(u_{sup}(t)) \ge -C(\varphi_\epsilon(t) - u_{sup}(t)) = -C\Phi_\epsilon(t)$$

where C_ϵ is a local Lipschitz constant for F in the interval $\{\varphi_\epsilon(s) : 0 \le s \le t\}$.

Then by integrating this equation we find that $\Phi_\epsilon(t) \ge \Phi_\epsilon(0)e^{-Ct} = \epsilon e^{-Ct}$ and so in particular $\Phi_\epsilon(\bar{t}) \ge \epsilon e^{-C\bar{t}} > 0$ but that contradicts the fact that $\Phi_\epsilon(\bar{t}) = 0 \longrightarrow\!\longleftarrow$.

So $u_{sup}(t) \le \varphi_\epsilon(t)$ for every $t \in [0,T_\delta)$ and so if we let $\epsilon \to 0$ then we have that $u_{sup}(t) \le \varphi(t)$ for every $t \in [0,T_\delta)$. Since $\delta > 0$ was arbitrary, we have proven the desired result for $[0,T')$. $\square$

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